An Analytical Solution to the Microstrip Line Problem

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Abstract—**An analytical method to determine the capacitance of microstrip lines in an inhomogeneous medium is presented. The capacitance is expressed in terms of the solution of an infinite system of linear equations. The numerical examples included provide a very good approximation for cases of practical interest.**

I. Introduction

MICROSCOPIC transmission lines have been the subject to theoretical and experimental investigation for more than 30 years. Although the literature on this subject is extensive, analytical exact solutions, for the case when two different dielectrics are present, have received consideration only recently [13], [14].

It is well known that in the case of nonhomogeneous dielectrics microstrip lines do not support a TEM mode. However, for frequencies that are not too high (in the "quasi-static limit"), the propagation can be thought of as approximately TEM. In this paper we consider this quasi-TEM mode.

Let us first briefly review some theoretical methods relating to a microstrip line with two dielectrics. Wheeler [1] used approximate conformal mapping and an interpolation technique to calculate the capacitance of a mixed dielectric media microstrip. Sylvester [2] and Bryant and Weiss [3] treated the dielectric vacuum boundary by means of a dielectric Green's function. Yamashita and Mittra [4] presented an analysis based on a variational principle. Analyses of various planar transmission lines have been carried out in the spectral domain by Itoh and Mittra [5] and Itoh [6]. Poh *et al.* [7] considered the solution for the line capacitance of a microstrip line by means of a spectral-domain analysis method.

The present paper provides a new analytical method for determining the line capacitance of a microstrip line. The solution is exact but it is expressed by means of the solution of an infinite system of linear equations whose coefficients are the result of certain numerical quadratures. The analysis is carried out for the case of two dielectric substrates. Changes to include additional stratified layers are readily available using the transfer matrix method described in [8].

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Comparison of the results obtained by using the proposed formula with those obtained by exact formulas (available in particular cases) shows that in cases of practical interest it is sufficient to consider only the first two equations in the above-mentioned infinite set of linear equations.

II. Problem Formulation

The cross section of the shielded microstrip line to be analyzed in this paper is shown in Fig. 1. It consists of a conducting strip of zero thickness placed on a dielectric substrate (on the $[-b, b]$ segment of the O_x axis) between two parallel ground planes. The relative dielectric constants, ϵ_1 and ϵ_2, and the dielectric thicknesses, h_1 and h_2, are arbitrary. In particular, as $h_1 \to \infty$ we obtain the open microstrip.

We consider the solution of the microstrip problem in the "quasi-static" approximation, i.e., for the frequency range in which propagation may be regarded as approximately TEM. For the microstrip dimensions and substrate materials frequently used in integrated microwave circuit technology, the quasi-TEM approximation is valid in the range extending to the low gigahertz region. For higher frequencies, the solution can be taken as the basis for solving the full propagation problem.

In the quasi-TEM state, the electric field in domains D_1 and D_2 can be expressed with the aid of the electrostatic potentials $V^{(1)}(x,y)$ and $V^{(2)}(x,y)$. We write

$$V^{(1)}(x,y) = \int_0^\infty A(k) \frac{\sinh k(h_1 - y)}{\sinh kh_1} \cos(kx)\, dk \tag{1}$$

$$V^{(2)}(x,y) = \int_0^\infty A(k) \frac{\sinh k(h_2 + y)}{\sinh kh_2} \cos(kx)\, dk. \tag{2}$$

The functions $V^{(1)}(x,y)$ and $V^{(2)}(x,y)$ vanish on the planes $y = h_1$ and $y = -h_2$, respectively, and satisfy the potential continuity condition on the circuit plane $y = 0$. We also have

$$\int_0^\infty A(k) \cos(kx)\, dk = V_0, \qquad -b < x < b \tag{3}$$

V_0 being the potential of the strip.

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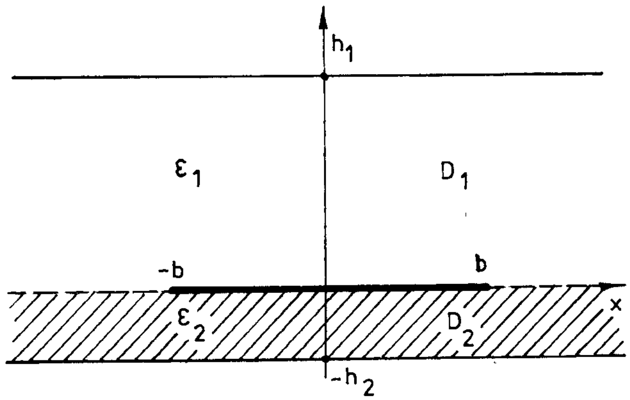

Fig. 1. Geometry of the problem.

The charge density $\rho(x)$ on the plane $y=0$ is given by the formula

$$\rho(x) = -\epsilon_1 \frac{\partial V^{(1)}}{\partial y}(x,0) + \epsilon_2 \frac{\partial V^{(2)}}{\partial y}(x,0) = \int_0^\infty kA(k)\{\epsilon_1 \coth(kh_1) + \epsilon_2 \coth(kh_2)\}\cos(kx)\,dk. \quad (4)$$

Since the function $\rho(x)$ vanishes on the semiaxis $(-\infty,-b)$ (b,∞) relation (4) yields

$$\int_0^\infty A(k)\{\epsilon_1 \coth(kh_1) + \epsilon_2 \coth(kh_2)\}\sin(kx)\,dk = \frac{q_0}{2}\operatorname{sgn} x, \qquad x \in (-\infty,-b)\cup(b,\infty) \quad (5)$$

where $\operatorname{sgn} x = -1$ for $x<0$ and $\operatorname{sgn} x = 1$ for $x>0$. In obtaining relation (5) we took into account the even character of the function $\rho(x)$. Hence

$$\int_{-b}^{b} \rho(x)\,dx = q_0 \quad (6)$$

where q_0 is the total charge of the strip.

Let us now put

$$A(k) = \frac{B(k)}{\epsilon_1 \coth(kh_1) + \epsilon_2 \coth(kh_2)} \equiv \frac{B(k)}{\epsilon_1+\epsilon_2}\{1-\eta(k)\} \quad (7)$$

Here

$$\eta(k) = \frac{\zeta(k)}{1+\zeta(k)} \quad (8)$$

$$\zeta(k) = \frac{2\epsilon_1}{\epsilon_1+\epsilon_2}\frac{e^{-2kh_1}}{1-e^{-2kh_1}} + \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\frac{e^{-2kh_2}}{1-e^{-2kh_2}}. \quad (9)$$

The functions $\zeta(k)$ and $\eta(k)$ decay exponentially for large values of the variable k.

Equations (3) and (4) become

$$\int_0^\infty B(k)\cos(kx)\,dk = (\epsilon_1+\epsilon_2)V_0 + \int_0^\infty B(k)\eta(k)\cos(kx)\,dk, \qquad -b<x<b \quad (10)$$

$$\int_0^\infty B(k)\sin(kx)\,dk = \frac{q_0}{2}\operatorname{sgn} x \qquad \text{for } x\in(-\infty,-b)\cup(b,\infty). \quad (11)$$

Relations (10) and (11) are the dual integral equations of the problem.

III. Solution of the Integral Equations

We look for the solution of the dual integral equations (10) and (11) in the form

$$B(k) = \frac{q_0}{\pi}\left\{\frac{J_0(bk)}{k} + 2c\delta(k)\right\} + \sum_{n=1}^{\infty} 2nb_n \frac{J_{2n}(bk)}{k} \quad (12)$$

where $\delta(k)$ is the Dirac function, $c=0.577216\ldots$ is the Euler constant, and b_n are real coefficients to be determined. In fact, the form (12) of the solution can be obtained by means of certain intricate mathematical considerations concerning the Riemann boundary-value problem equivalent to the dual integral equations (10) and (11).

We use now the relations

$$\int_0^\infty \left\{\frac{J_0(bk)}{k} + 2c\delta(k)\right\}\sin(kx)\,dk = \frac{\pi}{2}\operatorname{sgn} x \qquad \text{for } |x|>b \quad (13)$$

$$\int_0^\infty \left\{\frac{J_0(bk)}{k} + 2c\delta(k)\right\}\cos(kx)\,dk = \ln\frac{2}{b} \qquad \text{for } |x|<b \quad (14)$$

which are proved in the Appendix, and also the integrals

$$\int_0^\infty \frac{J_{2n}(bk)}{k}\sin(kx)\,dk = \begin{cases} \dfrac{1}{2n}\sin\left(2n\arcsin\dfrac{x}{b}\right) & \text{for } 0<x<b \\ 0 & \text{for } x>b \end{cases} \quad (15)$$

$$\int_0^\infty \frac{J_{2n}(bk)}{k}\cos(kx)\,dk = \begin{cases} \dfrac{1}{2n}\cos\left(2n\arcsin\dfrac{x}{b}\right) & \text{for } 0<x<b \\ \dfrac{(-1)^n}{2n}\dfrac{b^{2n}}{\left(x+\sqrt{x^2-b^2}\right)^{2n}} & \text{for } x>b \end{cases} \quad (16)$$

which are particular cases of the Weber–Schafheitlin integral [10, p. 743]. By means of these relations equation (11) is identically satisfied and relation (10) provides

$$\sum_{n=1}^{\infty} b_n \cos n\varphi = (\epsilon_1+\epsilon_2)V_0 - \frac{q_0}{\pi}\ln\left(\frac{2}{b}\right) + \int_0^\infty B(k)\eta(k)\cos\left(kb\sin\frac{\varphi}{2}\right)dk, \qquad \varphi\in(-\pi,\pi) \quad (17)$$

where $x = b\sin(\varphi/2)$ and $\varphi \in (-\pi, \pi)$. We use also the relation

$$\cos\left(kb\sin\frac{\varphi}{2}\right) = J_0(kb) + 2\sum_{m=1}^{\infty} J_{2m}(bk)\cos(m\varphi). \quad (18)$$

Relation (17) yields

$$(\epsilon_1+\epsilon_2)V_0 - \frac{q_0}{\pi}\ln\left(\frac{2}{b}\right) + \int_0^{\infty} B(k)\eta(k)J_0(bk)\,dk = 0 \quad (19)$$

$$b_m = 2\int_0^{\infty} B(k)J_{2m}(bk)\eta(k)\,dk. \quad (20)$$

Taking into account expression (20) for the coefficients b_m, relation (12) yields the following infinite system of linear equations:

$$b_m = q_0 a_m + \sum_{n=1}^{\infty} a_{mn}b_n, \qquad m = 1,2,\cdots. \quad (21)$$

Here we have denoted

$$a_m = \frac{2}{\pi}\int_0^{\infty}\frac{J_0(bk)J_{2m}(bk)}{k}\eta(k)\,dk, \qquad m = 1,2,\cdots \quad (22)$$

$$a_{mn} = 4n\int_0^{\infty}\frac{J_{2m}(bk)J_{2n}(bk)}{k}\eta(k)\,dk, \qquad m,n = 1,2,\cdots. \quad (23)$$

Relation (19) can be written in the form

$$(\epsilon_1+\epsilon_2)V_0 - \frac{q_0}{\pi}\ln\left(\frac{2}{b}\right) + \frac{1}{2}b_0 = 0 \quad (24)$$

where

$$b_0 = 2\int_0^{\infty} B(k)J_0(bk)\eta(k)\,dk. \quad (25)$$

By using expression (12) we get

$$b_0 = q_0 a_0 + \sum_{n=1}^{\infty} a_{0n}b_n. \quad (26)$$

Here

$$a_0 = \frac{2}{\pi}\left\{\int_0^1 \frac{J_0^2(bk)\eta(k)-1}{k}dk + \int_1^{\infty}\frac{J_0^2(bk)}{k}\eta(k)\,dk + c\right\} \quad (27)$$

and the coefficients a_{0n} are also defined by relation (23).

Relation (21) suggests to us to put $b_n = q_0\tilde{b}_n (n = 0,1,\cdots)$. Finally, formula (24) gives the strip capacitance C in the form

$$C = \frac{2(\epsilon_1+\epsilon_2)}{\frac{2}{\pi}\ln 2 - \tilde{b}_0} \quad (28)$$

TABLE I

h^*	0.075	0.1	0.125	0.250	0.5	1
C	14.22	10.883	8.8825	4.88254	2.882245	1.875541
C_0	12.25	9.775	8.1568	4.75261	2.870641	1.874973
C_1	14.54	10.905	8.8781	4.88248	2.882238	1.875541
C_2	14.56	10.907	8.8784	4.88254	2.882245	1.875541

where

$$\tilde{b}_0 = \tilde{a}_0 + \sum_{n=1}^{\infty}\tilde{a}_{0n}\tilde{b}_n \quad (29)$$

$$\tilde{b}_m = \tilde{a}_m + \sum_{n=1}^{\infty}\tilde{a}_{mn}\tilde{b}_n \qquad (m = 1,2,\cdots) \quad (30)$$

$$\tilde{a}_0 = \frac{2}{\pi}\left\{\int_0^1\frac{J_0^2(k)\eta(k/b)-1}{k}dk + \int_1^{\infty}\frac{J_0^2(k)}{k}\eta\left(\frac{k}{b}\right)dk + c\right\} \quad (31)$$

$$\tilde{a}_m = \frac{2}{\pi}\int_0^{\infty}\frac{J_0(k)J_{2m}(k)}{k}\eta\left(\frac{k}{b}\right)dk \qquad (m = 1,2,\cdots) \quad (32)$$

$$\tilde{a}_{mn} = 4n\int_0^{\infty}\frac{J_{2n}(k)J_{2m}(k)}{k}\eta\left(\frac{k}{b}\right)dk \qquad (m = 0,1,\cdots;\ n = 1,2,\cdots). \quad (33)$$

The effective evaluation of integrals (31)–(33) will be done numerically by using the Gauss–Laguerre formula.

IV. Numerical Results

In order to see how formula (28) works, we considered the particular case of the symmetrical structure ($h_1 = h_2 = h$). In this case a finite analytical expression for the line capacitance is available [9]:

$$\frac{C}{\epsilon_1+\epsilon_2} = 2\frac{K(k)}{K(k')}, \qquad k = \tanh\frac{\pi b}{2h}, k' = \sqrt{1-k^2} \quad (34)$$

$K(k)$ being the complete elliptical integral of the first kind.

In Table I we compared the exact values C of the capacitance given by formula (34) with approximate values C_0, C_1, and C_2 obtained by using formula (28) and the approximate solution of the infinite system (30) for various values of the ratio $h^* = h/(2b)$. We denoted by C_0 the value obtained by taking $\tilde{b}_0 = \tilde{a}_0$. C_1 stands for the approximate value resulting when a single (first) equation in the infinite system (30) is used ($\tilde{b}_j = 0$, $j > 1$), and C_2 is the capacitance following from relations (28) and (29) in the case where system (30) is truncated by considering only the first two equations ($\tilde{b}_0 = 0$, $j \geqslant 3$).

We remark that formula (28) for the case where

$$\tilde{b}_0 = \tilde{a}_0 + \frac{\tilde{a}_{01}\tilde{a}_1}{1-\tilde{a}_{11}} \quad (35)$$

yields approximate values for the capacitance with an error of about 2% for $h^* = 0.075$, 0.2% for $h^* = 0.1$, 0.05% for $h^* = 0.125$, and 0.001% for $h^* = 0.25$. From a practical viewpoint formulas (28) and (35) are sufficient for determining the strip capacitance for cases where $\min(h_1, h_2) > 0.15b$.

V. Conclusions

This paper has presented an analytical method for studying microstrip transmission lines in a quasi-static approximation.

The capacitance of the microstrip line is expressed in terms of the solution of an infinite system of linear equations. The numerical examples considered in the paper have pointed out that in practical cases it is sufficient to consider only the first equation in the system. This gives an approximate formula for the capacitance by substituting the value of $\tilde{b}_0$ given by formula (35) into relation (28).

Appendix

Let us determine the Fourier transform of the function

$$\frac{J_0(bk)}{|k|} + 2c\delta(k). \tag{A1}$$

Here the distribution sense is assumed [11], [12]. We have

$$\mathscr{F}\left\{\frac{J_0(bk)}{2|k|} + c\delta(k)\right\} = \mathscr{F}\left\{\frac{J_0(bk)-1}{2|k|}\right\} + \mathscr{F}\left\{\frac{1}{2|k|}\right\} + c \tag{A2}$$

where

$$\int_{-\infty}^{\infty} \frac{1}{|k|}\cdot\varphi(k)\,dk = \int_{|k|<1} \frac{\varphi(k)-\varphi(0)}{|k|}\,dk + \int_{|k|>1} \frac{\varphi(k)}{|k|}\,dk \tag{A3}$$

Here $\varphi(k)$ is a test function and $\mathscr{F}\{\ \}$ denotes the Fourier transform.

We also have [10], [12]

$$\int_0^\infty \frac{J_0(bk)-1}{k} e^{i\omega k}\,dk = \ln\frac{2\omega}{\omega + \sqrt{\omega^2+b^2}} \tag{A4}$$

$$\mathscr{F}\left\{\frac{1}{2|k|}\right\} = -\ln|\omega| - c. \tag{A5}$$

Relations (A2), (A4), and (A5) give

$$\int_0^\infty \left\{\frac{J_0(bk)}{k} + 2c\delta(k)\right\}\cos k\omega\,dk = \ln\frac{2}{b} \qquad \text{for } |\omega| < b. \tag{A6}$$

By using relation (A4) we also get

$$\int_0^\infty \left\{\frac{J_0(bk)}{k} + 2c\delta(k)\right\}\sin k\omega\,dk = \begin{cases} \dfrac{\pi}{2}\operatorname{sgn}\omega & \text{for } |\omega| < b \\ \arcsin\dfrac{\omega}{b} & \text{for } |\omega| > b. \end{cases} \tag{A7}$$

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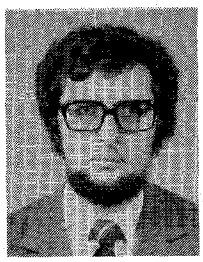

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